Journal of Engineering Mathematics **26**: 445–454, 1992.

Discussion related to Watson and Evans: "Resonant frequencies of a fluid in containers with internal bodies"

A.M.J. DAVIS
Mathematics Department, University of Alabama Tuscaloosa, AL 35487-0350, U.S.A.

Received 5 November 1990; accepted in revised form 10 March 1992

Abstract. Some recent authors have studied sloshing frequencies of fluid in a container, but have ignored the well-established literature on the asymptotic structure of the modes of linear oscillations. The present discussion draws attention to the more efficient solution methods thus suggested and relates the results to known asymptotic forms.

1. Introduction

The study of the resonant frequencies of oscillation of a fluid, possibly unbounded, with a free surface and rigid boundaries has a history that extends over more than two centuries, having been studied by Euler in 1761. The illustrious list of contributors also includes Poisson, Green, Airy, Stokes, Rayleigh, Kirchhoff and Poincaré. References to these and other authors can be found in the excellent review by Fox & Kuttler [1]. For a broader view of the subject, the reference list includes two papers [2, 3] that present distinctly different discussions of nonlinear resonant frequencies. Very few exact solutions are known but those for the rectangular basin in two dimensions and for the circular cylinder of uniform depth in three dimensions, display clearly an important feature of the high frequency modes. The free surface condition mandates a rapid exponential decay with depth and hence such oscillations are essentially confined to within a thin layer below the surface. Evidently, this physical feature is likely to be common to all geometries and, for two-dimensional sloshing frequencies, three conjectures are offered by Fox & Kuttler. Their first conjecture can be deduced from the other two which are stated as follows:

C2. The leading term in the asymptotic formula for the sloshing eigenvalues is independent of the shape or depth of the bottom or the area of the container, but depends only on the width of the free surface and the angles at which the sidewalls intersect the free surface.

C3. The eigenvalues of a region with sloshing surface $-1 \leqslant x \leqslant 1$, whose sidewalls intersect the surface with interior angles $\alpha\pi$, have the asymptotic form

$$\lambda_n \sim \left(n + \frac{1}{2} - \frac{1}{4\alpha}\right)\frac{\pi}{2}\,. \tag{1.1}$$

C2 was established by Davis [4] for vertical intersections by arguments including conformal mapping that can be readily extended to domains with $\alpha \neq \frac{1}{2}$, particularly now that McIver [5] has importantly solved the sloshing problem for a symmetric family of containers with $0 < \alpha < 1$.

C3 is verified not only by the elementary vertical walls case ($\alpha = \frac{1}{2}$) but also for the infinite dock with gap ($\alpha = 1$). Prior to the calculations of Miles [6], the two-dimensional eigenfrequencies were computed by Davis [7] who also established the asymptotic form and three error terms by using the known semi-infinite dock potential to construct an asymptotic solution of sufficient accuracy for high frequency sloshing modes in the presence of an infinite dock with gap.

McIver's solution, in terms of bipolar coordinates, is for a circular container filled to various depths determined by α. His results, for $\cos \alpha\pi$ decreasing from 0.8 to -1 in increments of 0.2, are not displayed in a form that allows easy comparison with (1.1). The length scale must be changed from the container radius to the semi-width of the free surface which varies with depth. Table 1 shows a selection of McIver's computed values, appropriately rescaled, and, for comparison, the corresponding values of (1.1). The evidence for the validity of C3 appears to be convincing. Moreover, since the leading term is almost immediately established in Davis' analysis, the asymptotic form (1.1) for the general case may be similarly deduced by noting the appearance of the phase angle $(1 - 1/2\alpha)\pi/4$ in the terms that dominate the surface disturbance in the standing wave sloping beach solutions given by Alker [8], after extraction from the work of Peters [9].

For sloshing in a half space bounded by a rigid plane with a circular hole, Troesch & Troesch [10] conjectured the asymptotic form

$$\lambda_n^m \sim \pi\left(n - \frac{1}{8} + \frac{1}{2}\,|m-1|\right)$$

for the nth eigenvalue in the mth azimuthal mode. By reference to the zeros of J_m', the corresponding result for a circular cylinder of uniform depth is

$$\lambda_n^m \sim \pi\left(n - \frac{1}{4} + \frac{1}{2}\,|m-1|\right).$$

Table 1. A selection of McIver's results, rescaled and compared with (1.1), for sloshing in a circular container filled so that $\alpha\pi$ is the angle subtended by the fluid at the free surface

	McIver results			Asymptotic estimates	
$\cos \alpha\pi$	Antisymmetric	Symmetric	$\alpha\pi$	Antisymmetric	Symmetric
0.8	0.62631	1.75745	0.6435	0.439025	2.00982
	3.31299	4.81815		3.58062	5.15141
	6.46034	8.09302		6.72221	8.29301
	9.70788	11.3262		9.86380	11.4346
0.4	1.06561	2.64803	1.159	1.29200	2.86280
	4.30640	5.92127		4.43359	6.00439
	7.51428	9.09742		7.57518	9.14598
	10.6765	12.2530		10.7168	12.2876
−0.2	1.47705	3.15142	1.772	1.66004	3.23083
	4.75290	6.33680		4.80163	6.37242
	7.91512	9.49069		7.94322	9.51402
	11.0650	12.6383		11.0848	12.6556
−0.8	1.81284	3.37616	2.498	1.86234	3.43313
	4.98833	6.54367		5.00393	6.57473
	8.13573	9.69514		8.14552	9.71632
	11.2798	12.8420		11.28711	12.8579

Table 2. A selection of McIver's results, rescaled and compared with (1.2), for sloshing in a spherical container filled so that $\alpha\pi$ is the angle subtended by the fluid at the free surface

	McIver results			Asymptotic estimates	
$\cos \alpha\pi$	$m = 0$	$m = 1$	$m = 2$	$m = 0, 2$	$m = 1$
0.8	2.29567	0.64339	1.26475	2.79522	1.22442
	5.55368	3.72049	5.03714	5.93681	4.36600
	8.85337	7.12927	8.57666	9.07840	7.50761
	12.0713	10.4153	11.8854	12.2200	10.6492
0.4	3.34541	1.15711	2.18834	3.64819	2.07740
	6.65936	4.92015	6.31176	6.78979	5.21899
	9.84794	8.19530	9.63091	9.93138	8.36058
	13.0112	11.3861	12.8511	13.0730	11.5022
−0.2	3.85855	1.75205	3.08555	4.01623	2.44543
	7.07296	5.38200	6.77484	7.15782	5.58703
	10.2410	8.60190	10.0437	10.2994	8.72862
	13.3965	11.7779	13.2476	13.4410	11.8702
−0.8	4.05851	2.37558	3.78928	4.21853	2.64774
	7.26834	5.67209	7.11493	7.36012	5.78933
	10.4376	8.85290	10.3261	10.5017	8.93092
	13.5942	12.0134	13.5056	13.6433	12.0725

It is now easy to conjecture that, for an axisymmetric basin that intersects the free surface at an interior angle $\alpha\pi$, the sloshing mode eigenvalues have asymptotic form

$$\lambda_n^m \sim \pi\left(n - \frac{1}{8\alpha} + \frac{1}{2}\,|m-1|\right). \tag{1.2}$$

Table 2 shows that this new conjecture is consistent with McIver's computations, based on a solution in terms of toroidal coordinates for a partially filled spherical container and rescaled as in Table 1.

2. Efficient solutions based on known asymptotics

Evans & McIver [11] and Watson & Evans [12] have extended the study to include containers that have another body either attached to a rigid wall or fixed in the free surface. The solution method employed by both sets of authors involves a complete set of vertical oscillatory eigenfunctions and hence cannot accurately approximate the underlying structure of sloshing modes in the presence of horizontal and vertical boundaries, namely, horizontal oscillations and exponential dependence on the vertical coordinate. These authors also made no reference to established or reliably conjectured asymptotic forms for the sloshing frequencies. Since these are known to differ only by exponentially small terms from the sloshing frequencies of the rectangular basin, it is apparent that any helpful calculation needs to have high accuracy and that the Fourier terms associated with the rectangle will provide, on matching along a submerged horizontal line, a matrix with pronounced diagonal dominance, particularly in the part to be truncated. Thus the high frequency modes play an important role in choosing an efficient truncation of the infinite matrix.

The basic problem considered by Watson & Evans has a rectangular block either attached to the bottom (case I) or fixed in the surface (case II) of a rectangular container. For two-dimensional sloshing modes of small amplitude, choose Cartesian coordinates x, y with y vertically downwards and $y = 0$ the undisturbed free surface. The rigid fluid container is set at $x = \pm b$, $y = d$ and the block occupies $|x| \leq a < b$, $0 < h \leq y \leq d$ (case I) or $0 \leq y \leq h < d$ (case II). The standard linearized theory of water waves permits the introduction of a velocity potential that, for periodic motion of radian frequency ω, takes the form

$$\Phi(x, y, t) = \mathrm{Re}\, \phi(x, y)\, \mathrm{e}^{-\mathrm{i}\omega t} ,$$

where $\nabla^2\phi = 0$ in the fluid,

$$K\phi + \frac{\partial\phi}{\partial y} = 0 \quad \text{at the free surface,} \tag{2.1}$$

with $K = \omega^2/g$, and ϕ has zero normal derivative at all rigid boundaries. Thus is defined an eigenvalue problem for K in which the geometry ensures that the eigenfunctions are either even or odd functions of x.

Evidently the most efficient solution method is that which exploits the physical reality that the bottom block is confined to a region where the sloshing modes have exponentially small amplitude while the surface block separates regions of fluid whose only connection is via depths where amplitudes are again exponentially small. Matching on a submerged line, where all but the lowest modes are evanescent, then causes only minor changes to their frequencies.

Case I

For an even mode, a solution $\phi_s(x, y)$ is given by

$$\phi_s = \sum_{n=1}^{\infty} \left[A_n \cosh \frac{n\pi}{b}(h - y) + C_n \sinh \frac{n\pi}{b}(h - y) \right] \cos \frac{n\pi x}{b} \quad (0 < y < h,\ |x| < b) ,$$

$$\phi_s = \frac{1}{2} B_0 + \sum_{n=1}^{\infty} B_n \frac{\cosh n\pi(d - y)/(b - a)}{\cosh n\pi(d - h)/(b - a)} \cos n\pi\left(\frac{|x| - a}{b - a}\right) \quad (h < y < d,\ a < |x| < b) .$$

Continuity of ϕ_s within the fluid now yields

$$B_p = \sum_{n=1}^{\infty} A_n \alpha_{np} ,$$

where

$$\alpha_{np} = \begin{cases} \dfrac{2n(1 - a/b) \sin n\pi a/b}{\pi[p^2 - n^2(1 - a/b)^2]} & \text{if } p \neq n(1 - a/b) , \\ (-1)^{n-p} & \text{if } p = n(1 - a/b) . \end{cases} \tag{2.2}$$

Meanwhile, the continuity of $\partial\phi_s/\partial y$ within the fluid and its vanishing at the top surface of the block yield

$$C_q = \sum_{p=1}^{\infty} \Gamma_{qp} A_p \quad (q \geq 1) ,$$

where

$$\Gamma_{qp} = \frac{1}{q} \sum_{n=1}^{\infty} n\alpha_{pn}\alpha_{qn} \tanh n\pi\left(\frac{d-h}{b-a}\right) \quad (q \geq 1)$$

in which the series converges quickly since the terms are $O(n^{-3})$. When the free surface condition (2.1) is applied to ϕ_s, it follows that the eigenvalues $\{K_{2m}b/\pi, m \geq 1\}$ for even modes of oscillation are those of the infinite matrix $\underline{F}^{-1}\underline{E}$ whose elements are defined by

$$\left.\begin{aligned} E_{qp} &= q\left(\delta_{qp} \tanh \frac{q\pi h}{b} + \Gamma_{qp}\right) \\ F_{qp} &= \delta_{qp} + \Gamma_{qp} \tanh \frac{q\pi h}{b} \end{aligned}\right\} \quad (p, q \geq 1) . \tag{2.3}$$

Similarly, the eigenvalues $\{K_{2m}b/\pi, m \geq 1\}$ for odd modes of oscillation are those of the infinite matrix $\hat{\underline{F}}^{-1}\hat{\underline{E}}$ whose elements are defined by

$$\left.\begin{aligned} \hat{E}_{qp} &= \left(q - \frac{1}{2}\right)\left(\delta_{qp} \tanh\left(q - \frac{1}{2}\right) \frac{\pi h}{b} + \hat{\Gamma}_{qp}\right) \\ \hat{F}_{qp} &= \delta_{qp} + \hat{\Gamma}_{qp} \tanh\left(q - \frac{1}{2}\right) \frac{\pi h}{b} \end{aligned}\right\} \quad (p, q \geq 1) , \tag{2.4}$$

where

$$\begin{aligned} \hat{\Gamma}_{qp} = {} & \frac{4}{\pi^2}\left(1 - \frac{a}{b}\right)^2\left(p - \frac{1}{2}\right)\cos\left(p - \frac{1}{2}\right)\frac{\pi a}{b}\cos\left(q - \frac{1}{2}\right)\frac{\pi a}{b} \\ & \times \sum_{n=1}^{\infty} \frac{n \tanh[n\pi(d-h)/(b-a)]}{\left[n^2 - \left(p - \frac{1}{2}\right)^2\left(1 - \frac{a}{b}\right)^2\right]\left[n^2 - \left(q - \frac{1}{2}\right)^2\left(1 - \frac{a}{b}\right)^2\right]} . \end{aligned}$$

Case II

For an even mode, a solution $\phi_s^*(x, y)$ is given by

$$\phi_s^* = \sum_{n=1}^{\infty} \left\{A_n^* \cosh n\pi\left(\frac{h-y}{b-a}\right) + C_n^* \sinh n\pi\left(\frac{h-y}{b-a}\right)\right\} \cos n\pi\left(\frac{|x|-a}{b-a}\right)$$
$$(0 < y < h,\ a < |x| < b) ,$$

$$\phi_s^* = \frac{1}{2} B_0^* + \sum_{n=1}^{\infty} B_n^* \frac{\cosh n\pi(d-y)/b}{\cosh n\pi(d-h)/b} \cos \frac{n\pi x}{b} \quad (h < y < d,\ |x| < b) .$$

After matching at $y = h$, application of the free surface condition (2.1) to ϕ_s^* shows that the eigenvalues $\{K_{2m}^*(b-a)/\pi;\ m \geq 1\}$ for even modes of oscillation are those of the infinite matrix $(\underline{F}^*)^{-1}\underline{E}^*$ whose elements are defined by

$$E^*_{qp} = q\left(\delta_{qp} \coth \frac{q\pi h}{b-a} + \Gamma^*_{qp}\right)$$
$$(p, q \geqslant 1)\,, \qquad (2.5)$$
$$F^*_{qp} = \delta_{qp} + \Gamma^*_{qp} \coth \frac{q\pi h}{b-a}$$

where

$$\Gamma^*_{qp} = p \sum_{n=1} \frac{1}{n}\, \alpha_{np}\alpha_{nq} \coth\left[\frac{n\pi}{b}(d-h)\right] \quad (q \geqslant 1)\,.$$

Similarly, the eigenvalues $\{K^*_{2m-1}(b-a)/\pi;\ m \geqslant 1\}$ for odd modes of oscillation are those of the infinite matrix $(\underline{\hat{F}}^*)^{-1}\underline{\hat{E}}^*$, whose elements are defined by

$$\hat{E}^*_{qp} = \left(q - \frac{1}{2}\right)\left(\delta_{qp} \coth\left[\left(q - \frac{1}{2}\right)\frac{\pi h}{b-a}\right] + \hat{\Gamma}^*_{qp}\right)$$
$$(p, q \geqslant 1)\,, \qquad (2.6)$$
$$\hat{F}^*_{qp} = \delta_{qp} + \hat{\Gamma}^*_{qp} \coth\left[\left(q - \frac{1}{2}\right)\frac{\pi h}{b-a}\right]$$

where

$$\hat{\Gamma}^*_{qp} = \frac{4}{\pi^2}\left(1 - \frac{a}{b}\right)^2 p \sum_{n=1}^{\infty} \frac{\left(n - \frac{1}{2}\right)\cos^2\left(n - \frac{1}{2}\right)\frac{\pi a}{b} \coth\left[\left(n - \frac{1}{2}\right)\frac{\pi}{b}(d-h)\right]}{\left[\left(n - \frac{1}{2}\right)^2\left(1 - \frac{a}{b}\right)^2 - q^2\right]\left[\left(n - \frac{1}{2}\right)^2\left(1 - \frac{a}{b}\right)^2 - p^2\right]}\,.$$

Computed values of $\{K_m b/\pi;\ m \geqslant 1\}$ and $\{K^*_m(b-a)/\pi;\ m \geqslant 1\}$ are displayed in Tables 3 and 4 respectively, for various values of a/b, d/b, h/d, and are seen to be close to half integers for m odd or integers for m even, as expected from the pronounced diagonal dominance exhibited by the matrix elements. 20 equations and 20 terms in each series sufficed for all values given. Ursell [13] showed that the eigenvalues could be simply estimated by applying Green's theorem to ϕ and an approximate eigenfunction that exactly satisfies (2.1). His method shows that in these cases the mth eigenvalue differs from $\frac{1}{2}m$ by terms containing the factor $\exp(-K_m h)$ or $\exp(-K^*_m h)$. The matrices in (2.3–6) suggest that the exponential factor in the asymptotic error is $\exp(-2K_m h)$ or $\exp(-2K^*_m h)$ in cases I and II respectively.

It may be observed that if the block is placed asymmetrically in $-c < x < a$, where $|c| < a$, then each set of eigenvalues and eigenfunctions bifurcates into two sets which in case I have the same asymptotic form but in case II have asymptotic forms based on the lengths $(b-a)$ and $(b-c)$ with sloshing essentially confined to one side of the surface block.

A related three dimensional problem, studied by Watson and Evans [12], concerns the resonant frequencies of oscillation of a liquid completely filling a circular cylindrical container of depth d and radius b with a free surface radius $a < b$. The two dimensional results suggest that the finite values of b and d cause at most exponentially small changes to the asymptotic form of the eigenvalues of Ka for the special case of a fluid bounded by a rigid half-plane containing a circular hole, considered by Miles [6] and Troesch and Troesch [10].

Table 3. Values of $K_m b/\pi$ for sloshing in a rectangular container of width $2b$ and depth d with a bottom block of width $2a$ and height $(d-h)$

	$a/b=1/4$		$a/b=1/2$		$a/b=3/4$	
	Odd	Even	Odd	Even	Odd	Even
$d/b=1/2$						
$h/d=0.5$	0.21871	0.83910	0.19433	0.71825	0.18740	0.66124
	1.41723	1.93874	1.34679	1.92217	1.25381	1.85238
	2.47657	2.98592	2.45048	2.97042	2.42029	2.95899
	3.49173	3.99614	3.48540	3.99288	3.47902	3.98914
	4.49785	4.99828	4.49610	4.99788	4.49432	4.99704
$h/d=0.2$	0.10785	0.62543	0.084590	0.38431	0.078462	0.31188
	1.25329	1.69487	0.91015	1.53289	0.68914	1.18326
	2.26869	2.82368	2.09431	2.59493	1.75544	2.36431
	3.31130	3.87067	3.14297	3.72557	2.97696	3.57198
	4.38043	4.89780	4.27117	4.79570	4.14485	4.70077
	5.42699	5.93076	5.33540	5.87480	5.24921	5.79482
	6.45010	6.95960	6.39743	6.91448	6.32471	6.87319
$d/b=2$						
$h/d=0.5$	0.46567	0.99877	0.46009	0.99711	0.45868	0.99634
	1.49994	2.00000	1.49987	1.99999	1.49977	1.99999
	2.50000	2.00000	2.50000	3.00000	2.49992	3.00000
$h/d=0.2$	0.31057	0.94755	0.28497	0.88259	0.27890	0.85282
	1.48605	1.99136	1.46301	1.98819	1.43620	1.97689
	2.49783	2.98553	2.49516	2.99890	2.49228	2.99754
$h/d=0.1$	0.18875	0.81742	0.15997	0.63287	0.15269	0.56341
	1.41632	1.90824	1.26538	1.86626	1.12402	1.73158
	2.45286	2.96788	2.40200	2.92791	2.32832	2.89605
	3.47639	3.98837	3.45717	3.97554	3.43786	3.96236
	4.49150	4.99516	4.48431	4.99012	4.47691	4.98587

Table 4. Values of $K_m^*(b-a)/\pi$ for sloshing in a rectangular container of width $2b$ and depth d with a bottom block of width $2a$ and depth b

	$a/b=1/4$		$a/b=1/2$		$a/b=3/4$	
	Odd	Even	Odd	Even	Odd	Even
$d/b=1/2$						
$h/d=0.5$	0.46183	0.97251	0.49571	0.99820	0.50260	1.00018
	1.50025	2.00010	1.50063	2.00017	1.50000	2.00000
	2.50039	3.00016	2.50004	3.00001	2.50000	3.00000
$h/d=0.2$	0.49289	0.98993	0.51785	1.01886	0.51647	1.00926
	1.52077	2.01759	1.51388	2.00973	1.50264	2.00098
	2.51266	3.00963	2.50501	3.00313	2.50060	3.00019
	3.50629	4.00454	3.50167	4.00099	3.50005	4.00002
	4.50300	5.00210	4.50057	5.00032		
$d/b=2$						
$h/d=0.2$	0.51204	1.00323	0.50468	1.00072	0.50039	1.00001
	1.50065	2.04919	1.50005	2.00001	1.50000	2.00000
	2.50003	3.00001	2.50000	3.00000	2.50000	3.00000
$h/d=0.1$	0.52836	1.01731	0.51662	1.00892	0.50484	1.00078
	1.50829	2.00399	1.50226	2.00082	1.50006	2.00000
	2.50175	3.00081	2.50022	3.00007	2.50000	3.00000

On suppressing the time factor as above, the boundary value problem for $\phi(r, \theta, z)$ may be expressed in terms of suitably chosen cylindrical polar coordinates (r, θ, z) in the form

$$\nabla^2\phi = 0 \quad (r < b,\ 0 < z < d)\,,$$

$$K\phi + \frac{\partial \phi}{\partial z} = 0 \quad \text{at } z = 0\,, \quad r < a\,,$$

$$\frac{\partial \phi}{\partial z} = 0 \quad \text{at} \begin{cases} z = 0\,, & a < r < b\,, \\ z = d\,, & r < b\,, \end{cases}$$

$$\frac{\partial \phi}{\partial r} = 0 \quad \text{at } r = b\,, \quad 0 < z < d\,.$$

On writing

$$\phi = \sum_{m=0}^{\infty} \phi_m(r, z)\, \mathrm{e}^{\mathrm{i}m\theta}\,,$$

the adaptation of Miles' method to finite boundaries yields the integral equation, for each $m \geq 0$,

$$f_m(r) = Ka^{-1} \int_0^a g_m(r, \eta) f_m(\eta)\eta\, \mathrm{d}\eta \quad (0 \leq r \leq a)\,, \tag{2.7}$$

where

$$f_m(r) = \frac{\partial \phi_m}{\partial z}(r, 0) \quad (m \geq 0)\,, \quad \text{i.e.} \int_0^a f_0(r) r\, \mathrm{d}r = 0\,,$$

$$g_m(r, \eta) = 2\frac{a}{b} \sum_{p=1}^{\infty} \frac{k'_{mp} J_m(k'_{mp} r/b) J_m(k'_{mp}\eta/b)}{(k'^2_{mp} - m^2)[J_m(k'_{mq})]^2} \coth k'_{mp} d/b\,, \tag{2.8}$$

and $J_m(k'_{mp}) = 0$ $(p \geq 1)$.

In the infinite depth case, $d \to \infty$ and then, according to Sneddon ([14], section 2.2), the series in (2.8) can be summed to yield the kernel

$$[g_m(r, \eta)]_{d=\infty} = a \int_0^\infty J_m(kr) J_m(k\eta)\, \mathrm{d}k - \frac{2a}{\pi} \int_0^\infty \frac{K'_m(tb)}{I'_m(tb)} I_m(tr) I_m(t\eta)\, \mathrm{d}t$$

in which the first term is that used by Miles. When each integral equation in (2.7) is converted, as described by Miles, to the system of N equations

$$A_l^m = Ka \sum_n C_{ln}^m A_n^m\,,$$

the coefficients can now be shown, with $n_m = n + \frac{1}{2}(m-1)$, to be given by

$$\pi C_{ln}^m = 8(l_m n_m)^{1/2} \left\{ \frac{2}{[1 - 4(l-n)]^2[4(l_m + n_m) - 1]} - \int_0^\infty \frac{K'_m(tb/a)}{I'_m(tb/a)} I_{2l_m}(t) I_{2n_m}(t) \frac{\mathrm{d}t}{t^2} \right.$$
$$\left. + (-1)^{l+n} \sum_{p=1}^{\infty} \frac{\exp(-k'_{mp} d/b)}{\sinh(k'_{mp} d/b)} \frac{J_{2n_m}(k'_{mp} a/b) J_{2l_m}(k'_{mp} a/b)}{k'_{mp}[J'_m(k'_{mp})]^2 (k'^2_{mp} - m^2)} \right\}. \tag{2.9}$$

Here, the range of l and n is $[1, N]$ for $m \geqslant 1$ but, due to the compatibility condition, is $[2, N+1]$ for $m = 0$. So $l_m, n_m > 0$ and, in particular, the range of values of l_0, n_0 is identical to that of l_2, n_2. The second and third contributions to (2.9) evidently represent modifications, due to the container's finite extent and depth respectively, of the matrix constructed by Miles and observed to be the same for the cases $m = 0, 2$. Since the terms in the series are at most of order $\exp(-2k'_{m1}d/b)$, the influence of finite depth on the asymptotic form of the eigenvalues of Ka is seen to be exponentially small as in the two dimensional modes. Thus the effect of finite extent can most conveniently be considered by setting $d = \infty$ in (2.9) and using an IMSL routine to compute inverse eigenvalues of the $N \times N$ matrix $\underline{C}^m$ for $m = 0, 1, 2$. At least the lowest $\frac{1}{2}N$ values of Ka/π are then obtained to the accuracy displayed in Table 5, which shows the remarkable durability, as b is reduced towards a, of the asymptotic form $\lambda_n^m \sim \pi(n - \frac{1}{8} + \frac{1}{2}|m-1|)$ for the nth eigenvalue of Ka in the mth azimuthal mode when the sloshing is in a half space. The crucial importance of the shape of the rigid boundary at its intersection with the free surface is amply demonstrated and the transition, as b approaches a, to the asymptotic form

$$\lambda_n^m = k'_{mn} \sim \pi\left(n - \frac{1}{4} + \frac{1}{2}|m-1|\right)$$

must be late and rapid. The presentation of Watson & Evans' results does not allow examination of this limit nor does their method, involving matching at the cylinder $r = a$, take exact account of the important rim edge at $z = 0$, $r = a$.

Table 5. Values of Ka/π, for various b/a, for sloshing frequencies in the lowest three azimuthal modes of fluid in an infinitely deep container of radius b with a circular aperature of radius a

$b/a = 3$	$b/a = 2$	$b/a = 1.5$	$b/a = 1.2$	$b/a = 1.05$
		$m = 1$		
0.87347	0.84551	0.77959	0.68709	0.61593
1.87322	1.85297	1.80888	1.75591	1.72031
2.87362	2.85814	2.82303	2.77990	2.75009
3.87393	3.86177	3.83295	3.79620	3.76994
4.87414	4.86430	4.84023	4.80850	4.78515
5.87428	5.86611	5.84566	5.81801	5.79721
		$m = 0$		
1.31099	1.30329	1.28495	1.25789	1.23485
2.33638	2.32903	2.30990	2.28158	2.25832
3.34719	3.34105	3.32373	3.29706	3.27499
4.35323	4.34820	4.33316	4.30907	4.28873
5.35710	5.35293	5.33997	5.31854	5.30008
6.35980	6.35627	6.34506	6.32606	6.30939
		$m = 2$		
1.31035	1.29349	1.24003	1.13482	1.02727
2.33596	2.32243	2.28246	2.21926	2.16965
3.34689	3.33623	3.30404	3.25439	3.21618
4.35300	4.34448	4.31797	4.27668	4.24460
5.35692	5.34993	5.32771	5.29252	5.26478
6.36965	6.35379	6.33483	6.30437	6.28000

References

1. D.W. Fox and J.R. Kuttler, Sloshing frequencies. *Z. angew. Math. Phys.* 34 (1983) 668–696.
2. G.R. Ghanimati and P.M. Naghdi, Oscillations over basins of variable depth. *J. Fluid Mech.* 164 (1986) 359–381.
3. R.W. Yeung and C.-F. Wu, Nonlinear wave body motion in a closed domain. *Comp. & Fluids* 17 (1989) 351–370.
4. A.M.J. Davis, Two-dimensional oscillations in a canal of arbitrary cross-section. *Proc. Camb. Philos. Soc.* 61 (1965) 827–846.
5. P. McIver, Sloshing frequencies for cylindrical and spherical containers filled to an arbitrary depth. *J. Fluid Mech.* 201 (1989) 243–257.
6. J.W. Miles, On the eigenvalue problems for fluid sloshing in a half-space. *Z. angew. Math. Phys.* 23 (1972) 861–869.
7. A.M.J. Davis, Waves in the presence of an infinite dock with gap. *J. Inst. Math. Applic.* 6 (1970) 141–156.
8. G. Alker, The scattering of short surface waves by a cylinder. *J. Fluid Mech.* 82 (1977) 673–686.
9. A.S. Peters, The effect of a floating mat on water waves. *Comm. Pure Appl. Math.* 3 (1950) 319–354.
10. B.A. Troesch and H.R. Troesch, A remark on the sloshing frequencies for a half-space. *Z. angew. Math. Phys.* 23 (1972) 703–711.
11. D.V. Evans and P. McIver, Resonant frequencies in a container with a vertical baffle. *J. Fluid Mech.* 175 (1987) 295–307.
12. E.B.B. Watson and D.V. Evans, Resonant frequencies of a fluid in containers with internal bodies. *J. Eng. Math.* 25 (1991) 115–136.
13. F. Ursell, Short surface waves in a canal; dependence of frequency on curvature. *J. Fluid Mech.* 63 (1974) 177–181.
14. I.N. Sneddon, *Mixed Boundary Value Problems in Potential Theory*. North Holland, Amsterdam (1966).